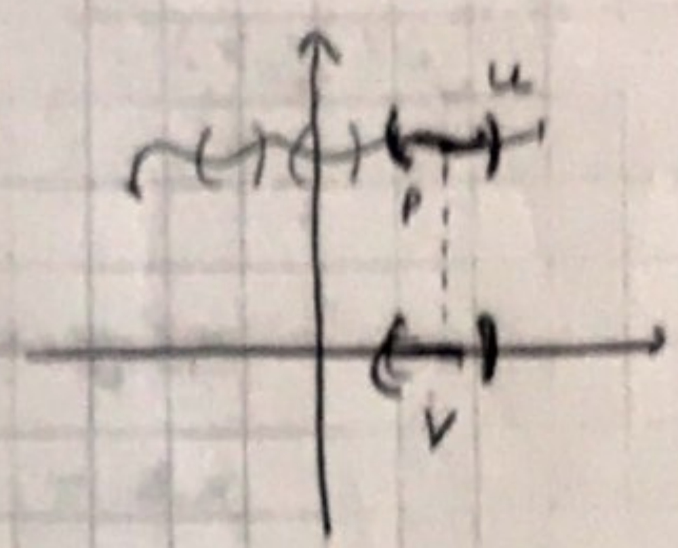
V 23.04. 2018. (XI nedelja)

Mnogostrukosti

Def. Za skup $M^n \subseteq \mathbb{R}^N$ (topološki prostor) $n \leq N$ kažemo da je n-dim mnogostrukost ako $\forall p \in M^n$ postoji okolina $U \ni p$, $U \subseteq M^n$ i postoji okolina $V \subseteq \mathbb{R}^n$ i $\exists$ $h : U \to V$ (homeomorfizam).

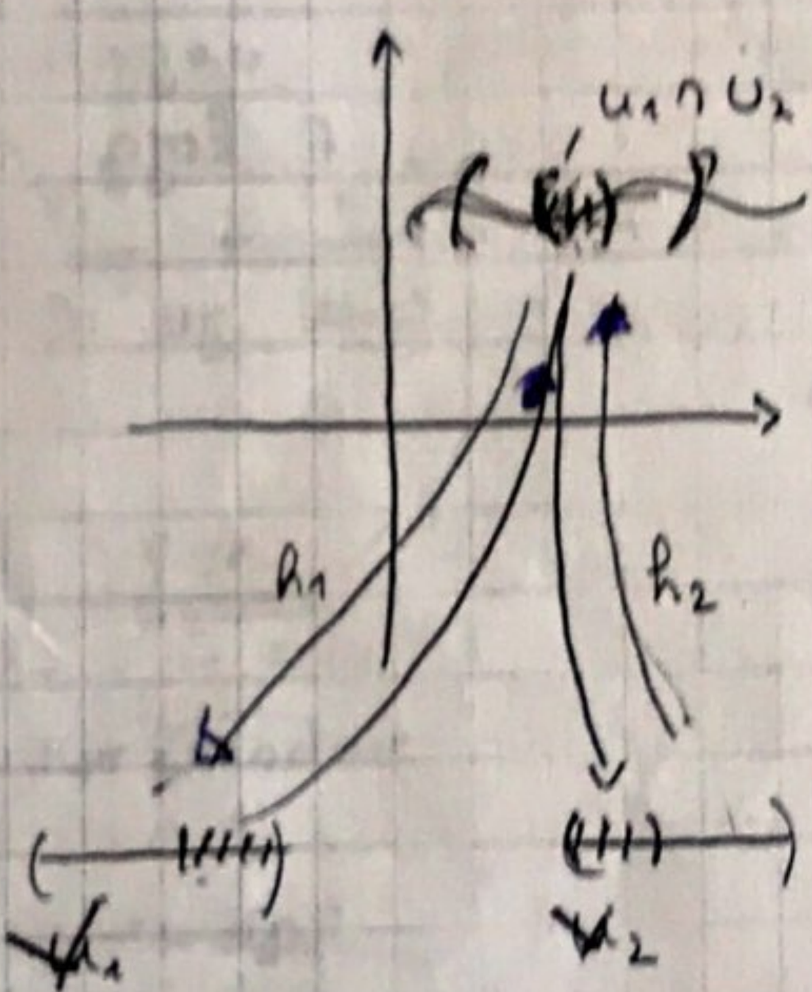


- kriva je mnogostrukost 1 u prostoru $\mathbb{R}^2$

(U, h) - karta mnogostrukosti

$\{(U_i, h_i)\}$ - atlas mnogostrukosti (skup svih karti date mnogostrukosti)

$$\bigcup_{i \in I} U_i = M^n$$



$$h_{12} = h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2)$$

↳ difeomorfizam

Def. Za n-dim mnogostrukost M^n kažemo da je glatka klase C^k ako $(\forall p \in M^n)(\exists (U, h), p \in U)$ $h : U \to V$, $\underline{g = h^{-1} : \underset{n\text{ dim}}{V} \to \underset{N\text{ dim}}{U}}$

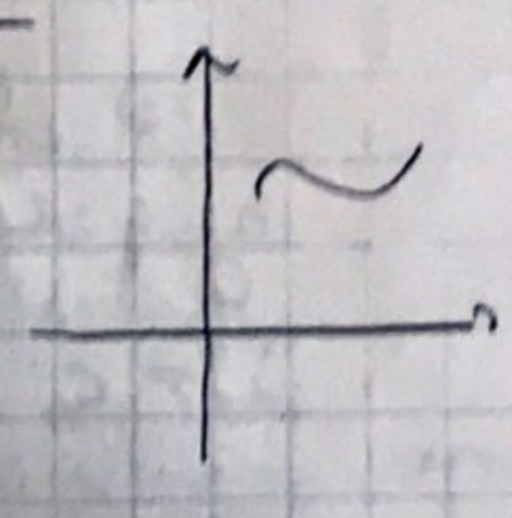
zadovoljava sledeće uslove: 1) $g \in C^k$

2) $\operatorname{rang}(g') = n$

primjer Grafik glatke f-je $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ je glatka mnogostruk

$\Rightarrow$ $\Gamma_f = \{(x_1, x_2, \dots, x_n, f(x_1, \dots, x_n)) \mid (x_1, \dots, x_n) \in U\} \subseteq \mathbb{R}^{n+1}$

Γ_f



$h : \Gamma_f \to V$ $(\subseteq \mathbb{R}^n)$ projekcija na prve n koordina

$\forall p \in \Gamma_f$ $p = (x_0^1, x_0^2, \dots, x_0^n, f(x_0^1, \dots, x_0^n$

$$h: \Gamma_f \to U \subseteq \mathbb{R}^n$$

$$(x_1, \dots, x_n, f(x_1, \dots, x_n)) \xrightarrow{h} (x_1, \dots, x_n)$$, h - homeomorfizam

Da bismo pokazali da je glatka mnogostrukost, moramo pokazati da njena inverzna f-ja $h^{-1} = g$ zadovoljava 2 uslova iz definicije:

$g(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n))$

1) $g \in C^k$ (ako je $f \in C^k$)

2)

$$g' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \\ f_{x_1} & \cdots & \cdots & f_{x_n} \end{pmatrix}_{(n+1) \times n} \begin{array}{l} \text{- za } x_1 \\ \text{- za } x_2 \\ \\ \\ \\ \end{array}$$

$\operatorname{rang} g' = n$

! (1.) Ekvipotencijalna površ $M = F^{-1}(\{c\})$ glatke f-je F koja je definisana u otvorenom skupu Ω sa <u>nenultim izvodom</u> je $(n-1)$-dim mnogostrukost.

D/

Pr. $x^2 + y^2 = 1$ ekvipotencijalna površ - kružnica

$F(x, y) = x^2 + y^2 - 1$

$J = (2x, 2y) \neq 0$ nenulti izvod

$\Rightarrow$ jeste mnogostrukost $\square$

$M = \{(x_1, \dots, x_n) \in \Omega \mid F(x_1, \dots, x_n) = c\} \subseteq \mathbb{R}^n$ ekvipot. površ

$(x_1^\circ, \dots, x_n^\circ)$ - fix tačku

(Treba da pokažemo da ima okolinu koju možemo slikati homeomorfno negdje u $\mathbb{R}^n$)

$dF \neq 0$

$\Rightarrow \exists i \; \frac{\partial F}{\partial x_i}(x_1^\circ, \dots, x_n^\circ) \neq 0$

Bez gubljenja opštosti, možemo reći $\frac{\partial F}{\partial x_n}(x_1^\circ, \dots, x_n^\circ) \neq 0$

$\frac{\partial F}{\partial x_n}(x_1^\circ, \dots, x_n^\circ) \neq 0$

$G = F(x_1, \dots, x_n) - c = 0$

$F(\underbrace{x_1^\circ, \dots, x_{n-1}^\circ}_{a}, \underbrace{x_n^\circ}_{b}) - c = 0$ tačka $(x_1^\circ, \dots, x_n^\circ)$ sigurno ovo zadovoljava (uzeli smo je iz mnogostrukosti)

F - glatka (dato)

$\Rightarrow$ T o impl. f-ji $\Rightarrow \exists U(x_1^\circ, \dots, x_{n-1}^\circ)$ i $\exists V(x_n^\circ)$ i f-ja $g: U \to V$

t.d. $g(x_1^\circ, \dots, x_{n-1}^\circ) = x_n^\circ$ i $F(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) = c$

kad god $(x_1, \dots, x_{n-1}) \in U$.

$(x_1, \dots, x_{n-1}) \in U \xrightarrow{g^*} (x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1}))$

g^* - preslikavanje koje ima rang n-1 (dob. se kao grafik f-je t.j. svedi smo na prethodni zadatak)

- njeno inverzno preslikavanje jeste homeomorfizam
- našli smo neku okolinu te tačke $U \times V$ dobili za $(x_1^0, \dots, x_n^0)$

=> svi uslovi definicije mnogostrukosti su zadovoljeni

primjer

a) $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ (koristeći prethodnu teoremu)

1-dim. mnogostrukost

$F(x,y) = x^2 + y^2$ $\quad F: \mathbb{R}^2 \to \mathbb{R}$

$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 2y$ - nenulti izvod (ne može biti tačka $(0,0)$ jer nije tačka sa krive)

b) $S^2 = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$ $\quad F: \mathbb{R}^3 \to \mathbb{R}$

Analogno, 2-dim mnogostrukost

c) $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} = 1$ - elipsoid mnogostrukosti n-1 u n-dim prostoru

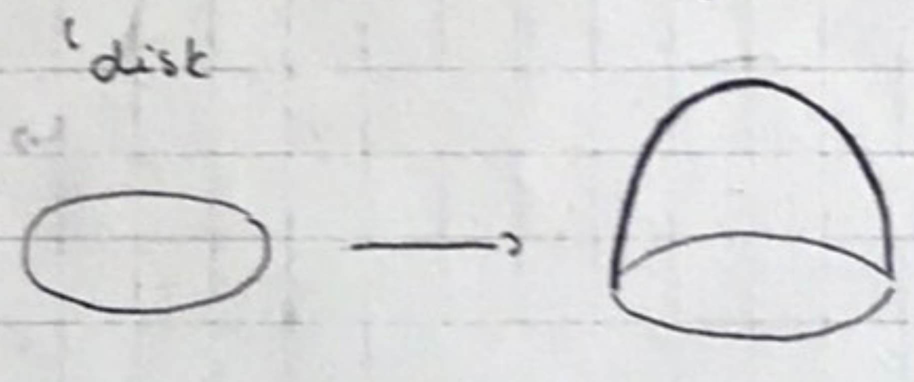
2. Sfera S^2 u $\mathbb{R}^3$

$h_1(x_1, x_2) = (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$

$h_1: D \to S^+$ (iz jediničnog diska u gornju polusferu)

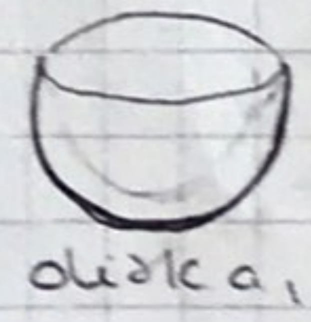
disk

- homeomorfizam



$h_2(x_1, x_2) = (x_1, x_2, -\sqrt{1 - x_1^2 - x_2^2})$ - homeomorf.

$h_2: D \to S^-$ (-11- u donju polusferu)



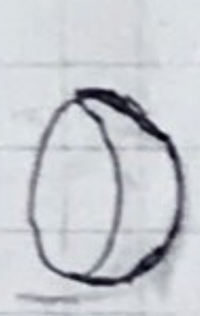
Želimo da napravimo da svaka tačka ima svoju okolinu.

Još su nam problem tačke sa granice diska, jer smo posmatrali otvoren skup

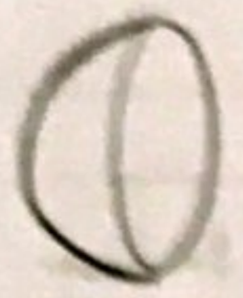
$h_3(x_1, x_3) = (x_1, \sqrt{1 - x_1^2 - x_3^2}, x_3)$

$h_3: D \to S^D$

desna polusfera



$h_4(x_1, x_3) = (x_1, -\sqrt{1-x_1^2-x_3^2}, x_3)$



$h_4 : D \to S^2$ lijeva polusfera

Ostale su nam još dvije tačke :

$h_5(x_2, x_3) = (\sqrt{1-x_2^2-x_3^2}, x_2, x_3)$

$h_6(x_2, x_3) = (-\sqrt{\text{--}''\text{--}}, x_2, x_3)$

Na ovaj način smo garantovali da svaka tačka ima svoju okolinu.

$\{(h_i(D), h_i)\}_{i=1}^{6}$ - atlas od 6 karti ; Sfera jeste mnogostrukost.

Cilj nam je da napravimo atlas sa manje karti.

$\Rightarrow$ Sferne koordinate (II način)

$x = \cos\varphi \sin\theta$

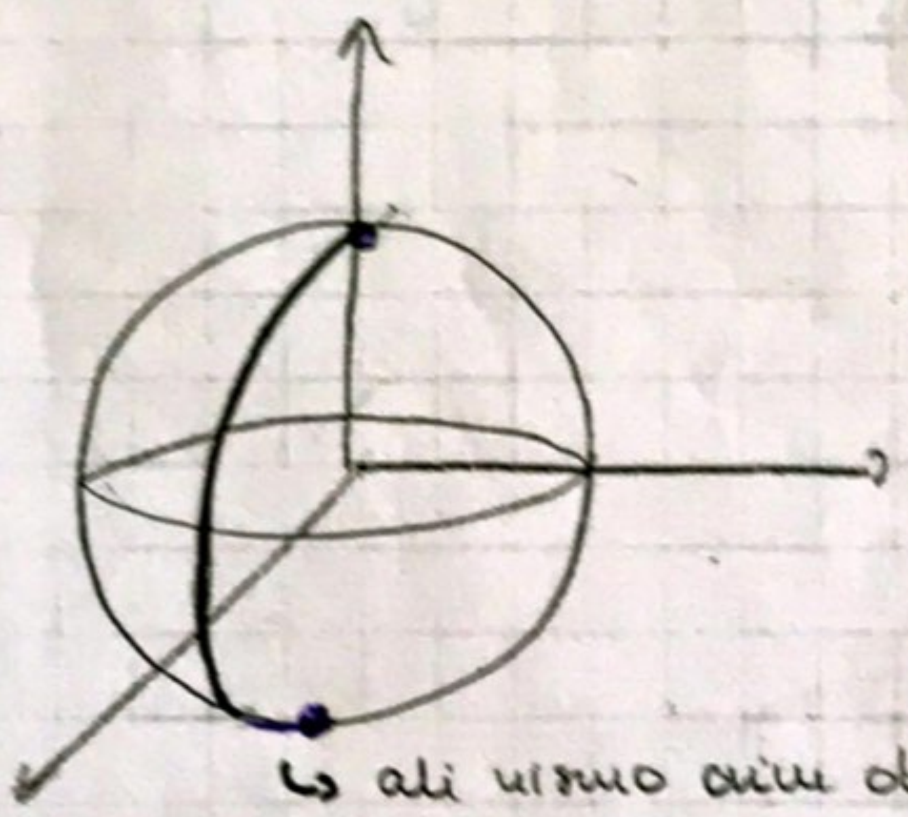
$y = \sin\varphi \sin\theta$

$z = \cos\theta$

1) $\varphi_1 \in (0, 2\pi)$ - otvoreni skupovi

$\theta \in (0, \pi)$

$(\varphi, \theta) \to (x, y, z)$

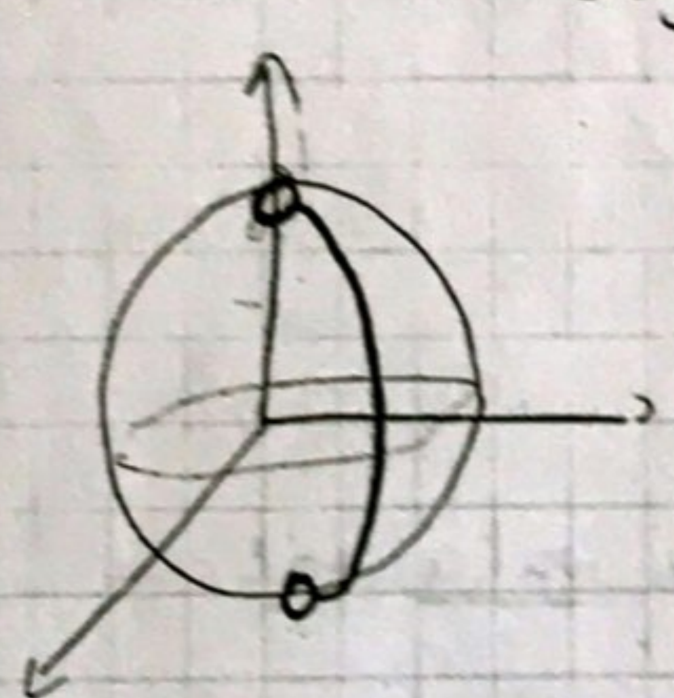


↳ ali nismo ovim obuhvatili ovaj meridijan

2) $\varphi_2 \in (\delta, 2\pi + \delta)$

$\delta > 0$ (proizvoljno malo)

$\theta \in (0, \pi)$



Još da riješimo θ i 0 i π - tačke koje nismo obuhvatili. (za $\theta = 0$, i $\theta = \pi$)

3) $\varphi_3 \in (0, 2\pi)$

$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

obuhvatiće gornju polusferu

4) $\varphi_4 \in (0, 2\pi)$

$\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$

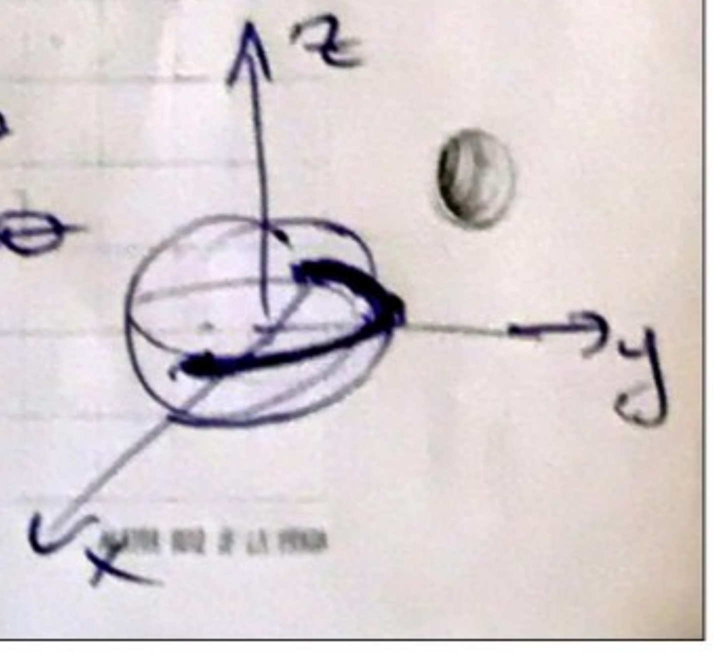
$x = \cos\theta$

$y = \cancel{\cos}\varphi \sin\theta$

$z = \sin\varphi \sin\theta$

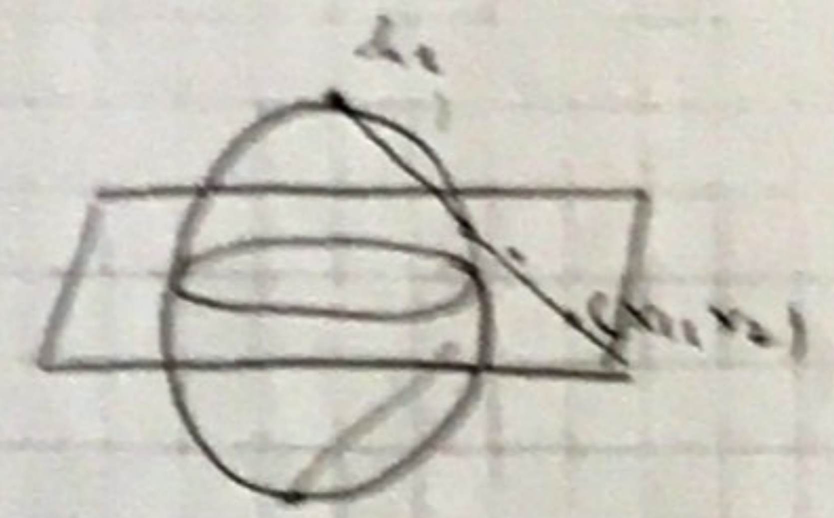
$\varphi \in (0, 2\pi)$

$\theta \in (0, \pi)$



$\Rightarrow$ 4 karte ukupno

Stereografska projekcija (III način)



- svaka tačka sem sjevernog pola se može slikati u $\mathbb{R}^2$, svaka tačka koja nije sjeverni pol će imati okolinu koja će se slikati u neku ravan

iz sj. pola, $h_1(x_1, x_2, x_3) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right)$; $(x,y) \xrightarrow{h_1^{-1}} \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\right)$

iz južnog pola, $h_2(x_1, x_2, x_3) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\right)$; $(x,y) \xrightarrow{h_2^{-1}} \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2}\right)$

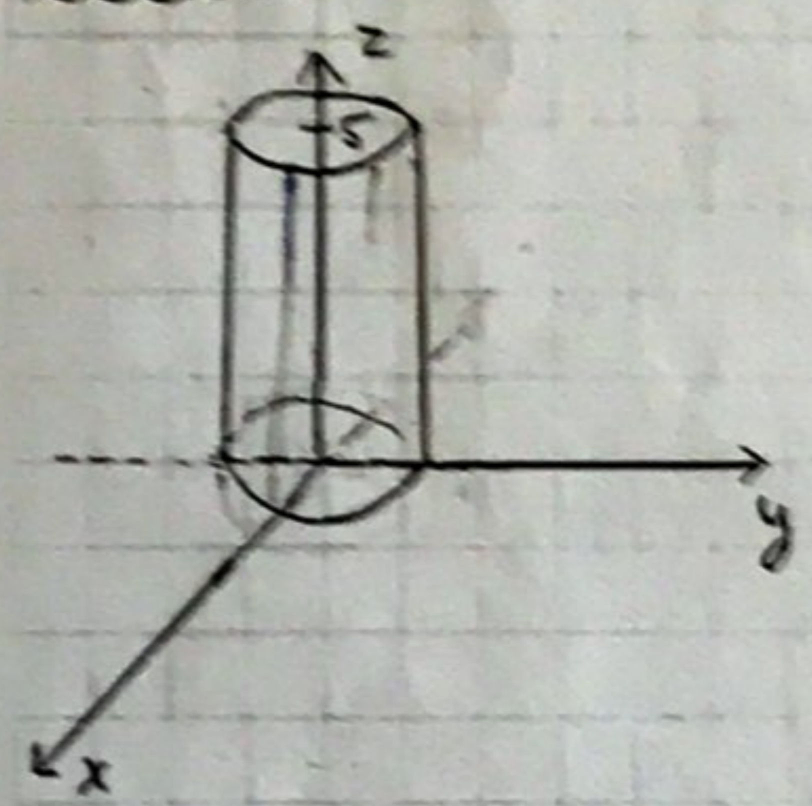
$\Rightarrow$ ukupno dvije karte.

$(x,y) \to \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$

$\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$

Zaključak: Ne postoji atlas od jedne karte (za sferu).

primjer $C = \{(x,y,z) \mid x^2+y^2 = 1, \ 0 < z < 5\}$



$(\varphi, z) = (\cos\varphi, \sin\varphi, z)$

1° $\varphi_1 \in (0, 2\pi)$

$z \in (0, 5)$

2° $\varphi_2 \in \left(\frac{\pi}{4}, 2\pi + \frac{\pi}{4}\right)$

$z \in (0, 5)$

$g(\varphi, z) = (\cos\varphi, \sin\varphi, z)$

$g' = \begin{pmatrix} -\sin\varphi & 0 \\ \cos\varphi & 0 \\ 0 & 1 \end{pmatrix}$; ako je $\sin\varphi \neq 0 \quad \vee \quad \cos\varphi \neq 0$

$\begin{pmatrix} -\sin\varphi & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \cos\varphi & 0 \\ 0 & 1 \end{pmatrix}$

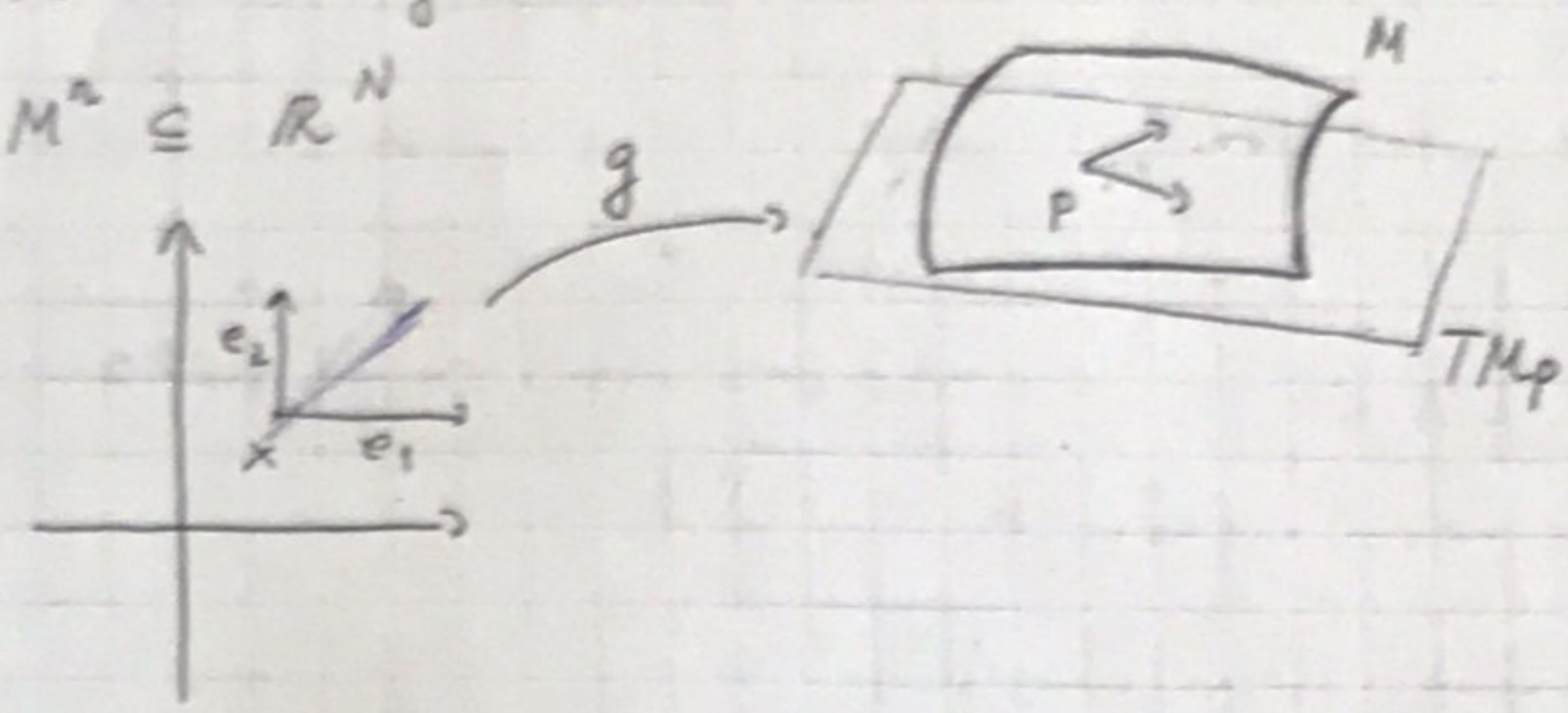
$\Rightarrow$ rang $= 2$

$\Rightarrow$ glatka mnogostrukost

Tangentni prostor

M^n - mnogostrukost

$M^n \subseteq \mathbb{R}^N$



$g(x) = p$

Baza za T_pM (tangentna ravan u tački p), $p = g(x)$

$((g'(x)e_1, p), \dots, (g'(x)e_n, p))$

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p$

p - u toj tački se nalazimo; tu smo napravili "centar"

$\dim T_pM = n$

⊙ Tangentni prostor ne zavisi od izbora baze.

$p = g(x)$

$g'(x)$

$v_p = \underbrace{g'(x)\,v}_{\text{ako imamo neki vek. } v \text{ (iz tang. ravni) tački } x} = (g'(x)v, p)$

$(v_1, p) + (v_2, p) = (v_1 + v_2, p)$

$$T_pM^n = \left\{ v_p \in \mathbb{R}^N_p \mid \exists v_x \in \mathbb{R}^n_x \text{ t.d. } v_p = g'(x)v_x \right\}$$

1. Odrediti koordinate vek. $v_p = (p, (v_1, v_2, v_3)) \in T_p S^2(0,1)$ u tački $p = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ koja se u (x,y) ravni vidi kao vektor $\underset{\sim v_x}{(1,1)}$.

R:

$g(x,y) = (x, y, \sqrt{1 - x^2 - y^2})$ - biramo gornju polusferu (jer sadrži ~~tačku (1,1)~~)

$g\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

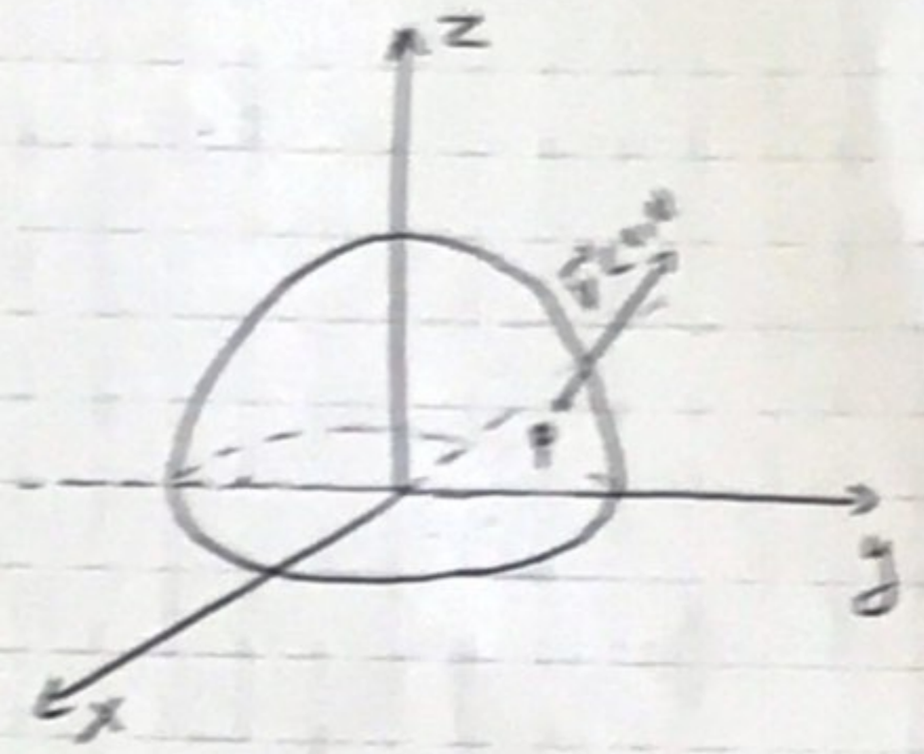
$$g'(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix}$$

$$f(x,y) = \sqrt{1-x^2-y^2}$$

$$f_x = \frac{-x}{\sqrt{1-x^2-y^2}} \xrightarrow{(\frac{1}{2},\frac{1}{2})} -\frac{1}{\sqrt{2}}$$

$$f_y = \frac{-y}{\sqrt{-''-}} \xrightarrow{(\frac{1}{2},\frac{1}{2})} -\frac{1}{\sqrt{2}}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{g'(x)} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v} = \begin{pmatrix} 1 \\ 1 \\ -\frac{2}{\sqrt{2}} \end{pmatrix}$$



② Odrediti j-nu tangentnog prostora $T_p M^2$ u mnog. 2, gdje je M^2: sfera, elipsoid, torus, konus.

$p(x_0, y_0, z_0) \in M^2$

$(x-x_0, y-y_0, z-z_0)$ - ovaj vek. da bude komplanaran sa vek. koji se odje pojavljuje

$\left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p$

$(x,y) \to (x, y, f(x,y))$

$$\begin{vmatrix} 1 & 0 & x-x_0 \\ 0 & 1 & y-y_0 \\ f_x & f_y & z-z_0 \end{vmatrix} = 0 \iff \text{komplanarnost}$$

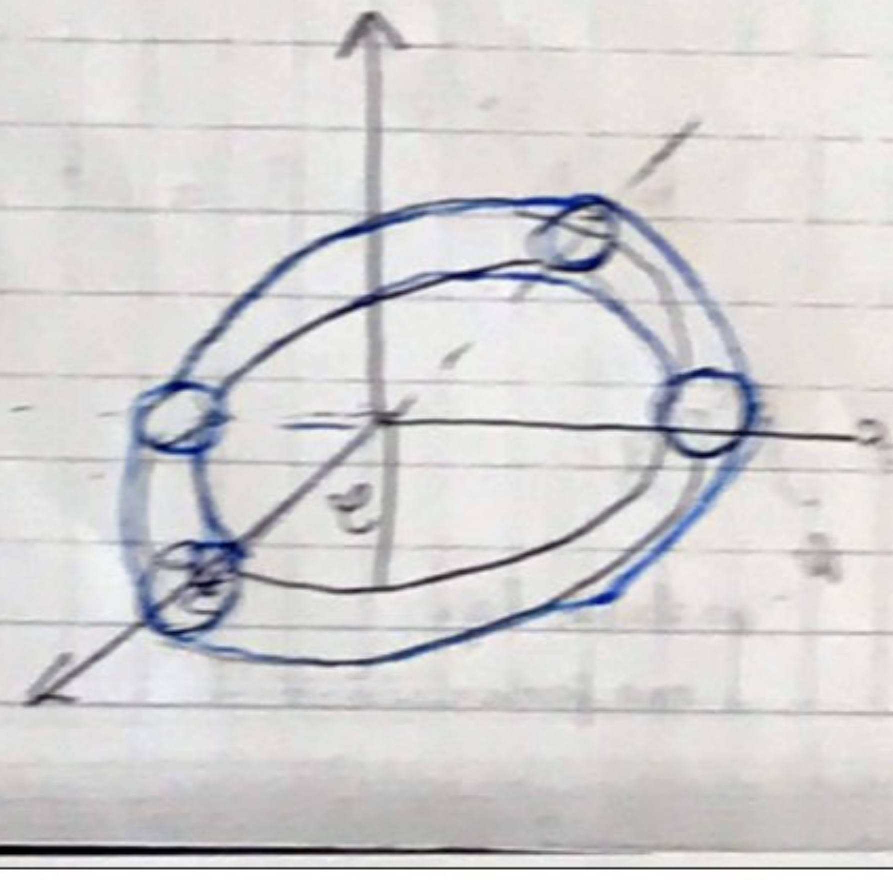
Torus

$x = (R + r\cos\theta)\cos\varphi$

$y = (R + r\cos\theta)\sin\varphi$

$z = r\sin\theta$

$\varphi, \theta \in (0, 2\pi)$



$(\varphi, \theta) \xrightarrow{g} (x, y, z)$

$$\frac{\partial g}{\partial \varphi} = \begin{pmatrix} -(R + r\cos\theta)\sin\varphi \\ (R + r\cos\theta)\cos\varphi \\ 0 \end{pmatrix} \xrightarrow[\theta = \frac{\pi}{4}]{\varphi = \frac{\pi}{2}} \begin{pmatrix} -(R + \frac{r}{\sqrt{2}}) \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial g}{\partial \theta} = \begin{pmatrix} -r\sin\theta\cos\varphi \\ -r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} \xrightarrow[\theta = \frac{\pi}{4}]{\varphi = \frac{\pi}{2}} \begin{pmatrix} 0 \\ -\frac{r}{\sqrt{2}} \\ \frac{r}{\sqrt{2}} \end{pmatrix}$$

Biramo neku tačku $\varphi = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$

Kad uvrstimo to φ i θ

$$\begin{array}{c} \begin{vmatrix} -(R + \frac{r}{\sqrt{2}}) & 0 & \omega_1 \\ 0 & -\frac{r}{\sqrt{2}} & \omega_2 \\ 0 & \frac{r}{\sqrt{2}} & \omega_3 \end{vmatrix} = 0 \\ \frac{\partial g}{\partial \varphi} \quad\quad \frac{\partial g}{\partial \theta} \end{array}$$

$\omega_1 = x - 0$

$\omega_2 = y - (R + \frac{r}{\sqrt{2}})$

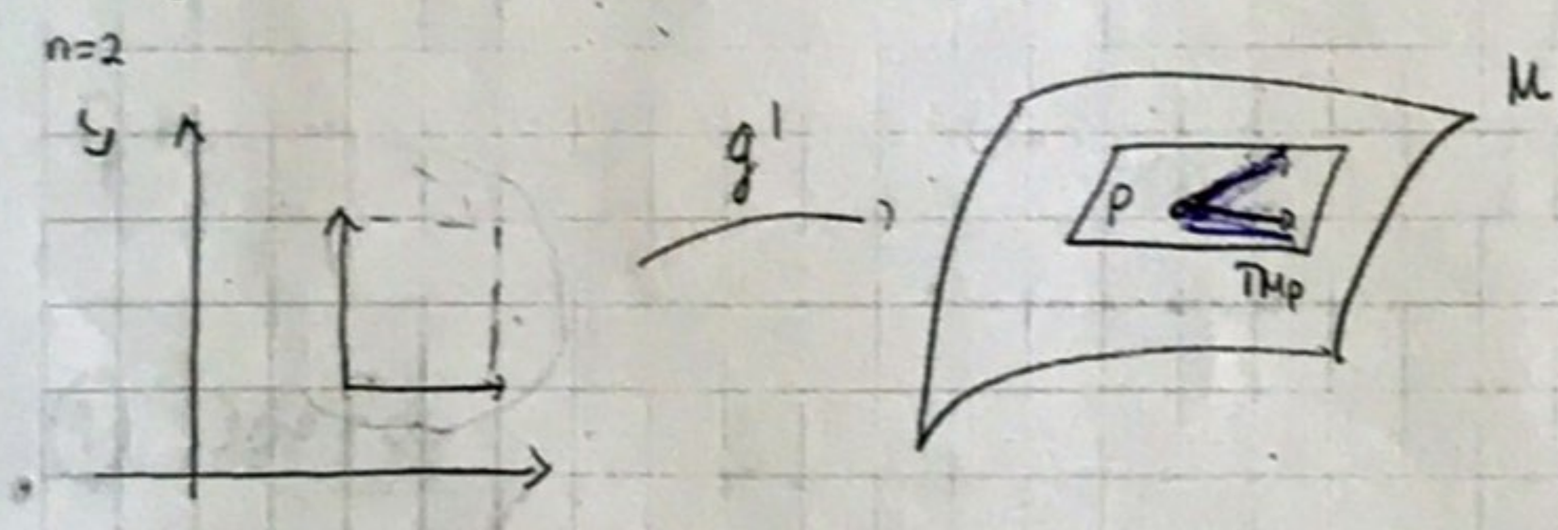
$\omega_3 = z - (\frac{r}{\sqrt{2}})$

ove tačke se dobiju kada $\varphi = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$ t.j. tačku koju smo izabrali uvrstimo u parametrizaciju koju smo napravili

$\omega_2 + \omega_3 = 0$ (i onda se dob tang ravan)... $\Rightarrow$ $\boxed{y + z = R + \frac{2r}{\sqrt{2}}}$

kada se izračuna det, dob. se...

Mjera skupa na mnogostrukosti. Integral po skupu mnogostrukosti.



$v_1, \dots, v_m \subseteq \mathbb{R}^n$

m paralelopiped m-zapeti

$\sqrt{\det(A^T A)}$ gdje je $A \in \mathbb{R}^{n \times m}$

$v_1 \quad v_2 \quad \dots \quad v_m$

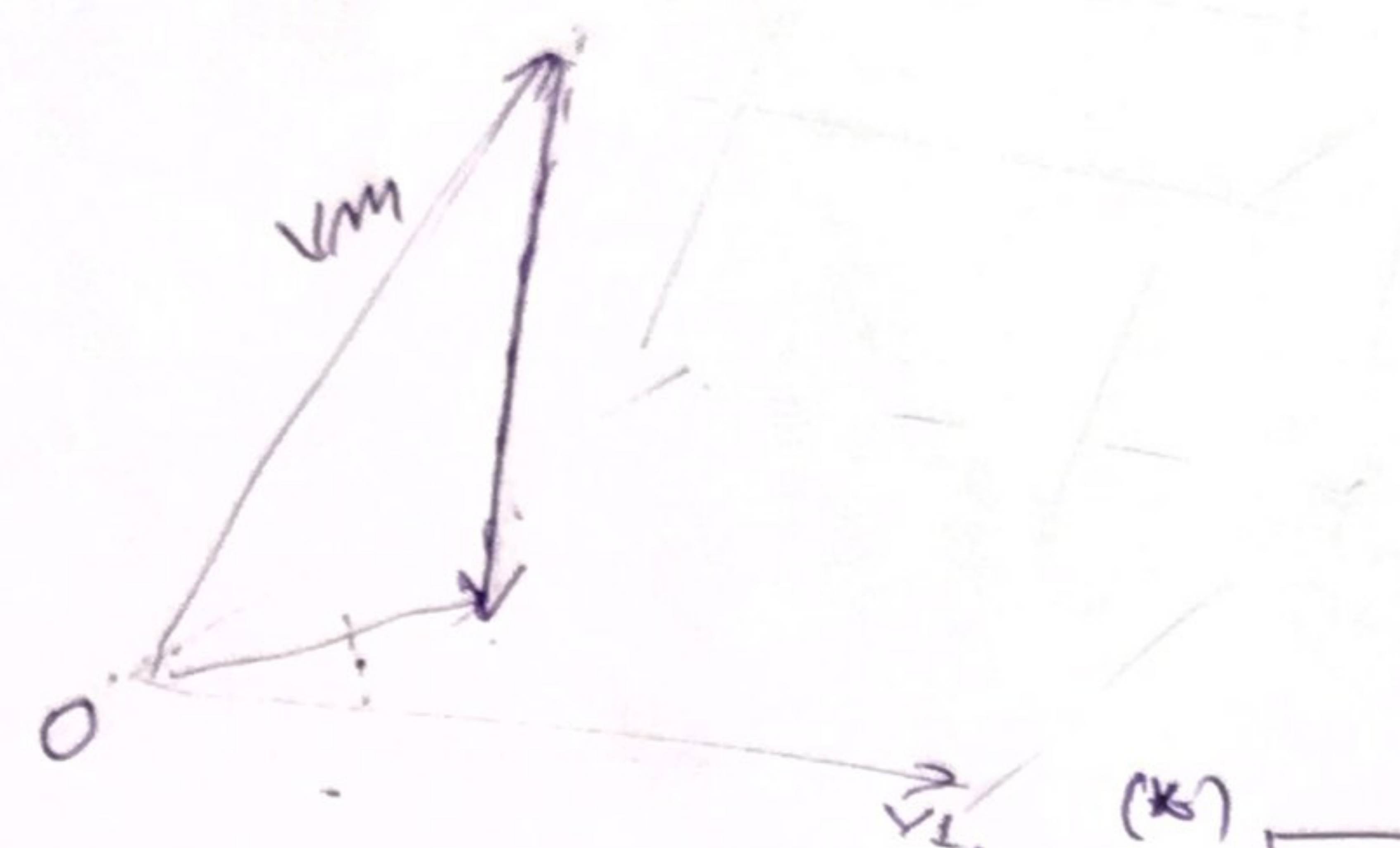
$\mu(\Pi_p) = \mu(I_x^n)\sqrt{g'}$, $\sqrt{g} = \sqrt{G((\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_n})_p)}$

$M^n = \{ g(x) \mid x \in D, \ D \subseteq \mathbb{R}^n \} \subseteq \mathbb{R}^N$ — otvoren skup

$\mu(M^n) = \int_D \sqrt{g}$

D ~ po skupu koji smo parametrizovali

$P(v_1, \dots, v_m) \subseteq \mathbb{R}^n$



Korak ind:

$$V_m(P) = V_m(v_1, \dots, v_{m-1}) \cdot h \overset{(*)}{=} \sqrt{G(v_1, \dots, v_{m-1})} \cdot h$$

$$\langle -v_m + \alpha_1 v_1 + \dots + \alpha_{m-1} v_{m-1}, v_j \rangle = 0 \quad , \quad j = \overline{1, m-1}$$

$$h^2 = \langle -v_m + \alpha_1 v_1 + \dots + \alpha_{m-1} v_{m-1}, -v_m + \alpha_1 v_1 + \dots + \alpha_{m-1} v_{m-1} \rangle$$

$$= \langle v_m, v_m \rangle - \alpha_1 \langle v_1, v_m \rangle - \dots - \alpha_{m-1} \langle v_{m-1}, v_m \rangle$$

$$\begin{cases} \underline{\alpha_1} \langle v_1, v_1 \rangle + \dots + \underline{\alpha_{m-1}} \langle v_{m-1}, v_1 \rangle + 0 \cdot \underline{h^2} = \langle v_1, v_m \rangle \\ \vdots \\ \underline{\alpha_1} \langle v_1, v_{m-1} \rangle + \dots + \underline{\alpha_{m-1}} \langle v_{m-1}, v_{m-1} \rangle + 0 \cdot \underline{h^2} = \langle v_{m-1}, v_m \rangle \\ \underline{\alpha_1} \langle v_1, v_m \rangle + \dots + \underline{\alpha_{m-1}} \langle v_{m-1}, v_m \rangle + 1 \cdot \underline{h^2} = \langle v_m, v_m \rangle \end{cases}$$

Kramerov metod

$$h^2 = \frac{G(v_1, \dots, v_{m-1}, v_m)}{G(v_1, \dots, v_{m-1})} \Rightarrow h = \frac{\sqrt{G(v_1, \dots, v_m)}}{\sqrt{G(v_1, \dots, v_{m-1})}}$$

$$\overset{(*)}{=} \sqrt{G(v_1, \dots, v_{m-1}, v_m)}$$

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$$\sqrt{g} = \begin{vmatrix} \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_1} \rangle & & \\ & \ddots & \\ & & \langle \frac{\partial g}{\partial x_n}, \frac{\partial g}{\partial x_1} \rangle \end{vmatrix}^{\frac{1}{2}} > 0$$

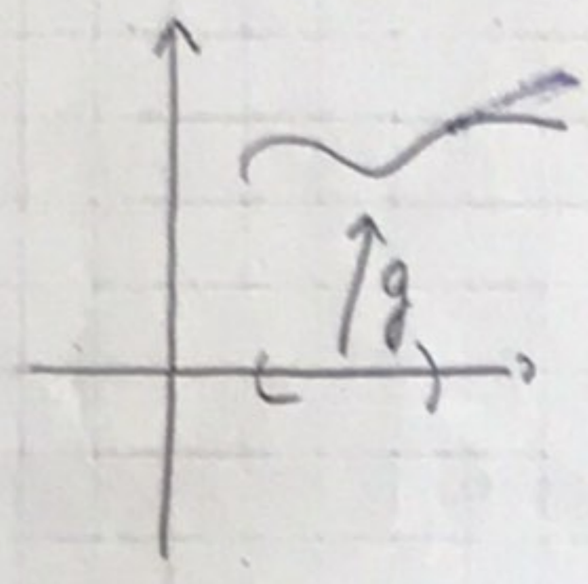
Integral po mnogostrukosti

$A \subseteq M^n$

(U, h) - karta

$$\int_A f = \int_{h(A)} f \circ h \sqrt{h}$$

Ako je $n = 1$

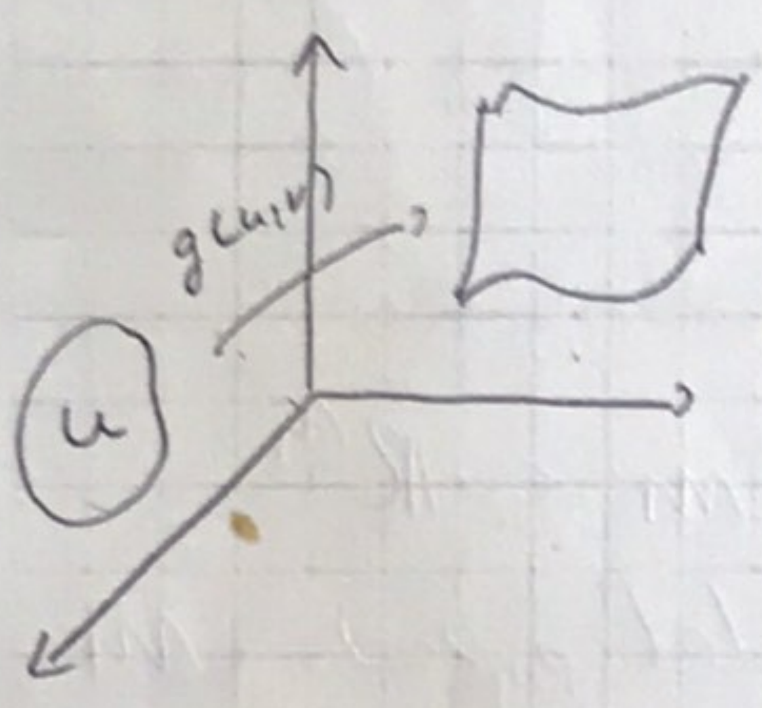


Dg:

$$\left| \langle \frac{\partial g}{\partial t}, \frac{\partial g}{\partial t} \rangle \right|^{\frac{1}{2}} = \left\| \frac{\partial g}{\partial t} \right\|$$

→ ako pravimo param, parametar će biti 1 (mnog. 1)

Ako je $n = 2$



Dg:

$$\begin{vmatrix} \langle g_u, g_u \rangle & \langle g_u, g_v \rangle \\ \langle g_v, g_u \rangle & \langle g_v, g_v \rangle \end{vmatrix}^{\frac{1}{2}} = \sqrt{EG - F^2}$$

$E = \langle g_u, g_u \rangle$

$F = \langle g_u, g_v \rangle$

$G = \langle g_v, g_v \rangle$

1) a) O kružnice polupr. R

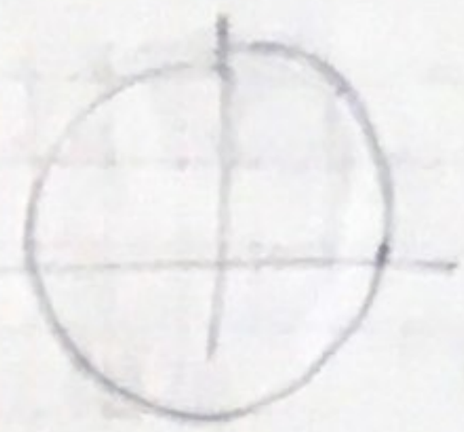
b) Naći dužinu zavojnice $x = R\cos t,\ y = R\sin t,\ z = ct$

$t \in (0, 2\pi)$

a) $g(t) = (R\cos t, R\sin t)$ ← parametrizujemo kružnicu

$\|g'(t)\| = R$

$\mu(S^1) = \int_0^{2\pi} R = 2\pi R$

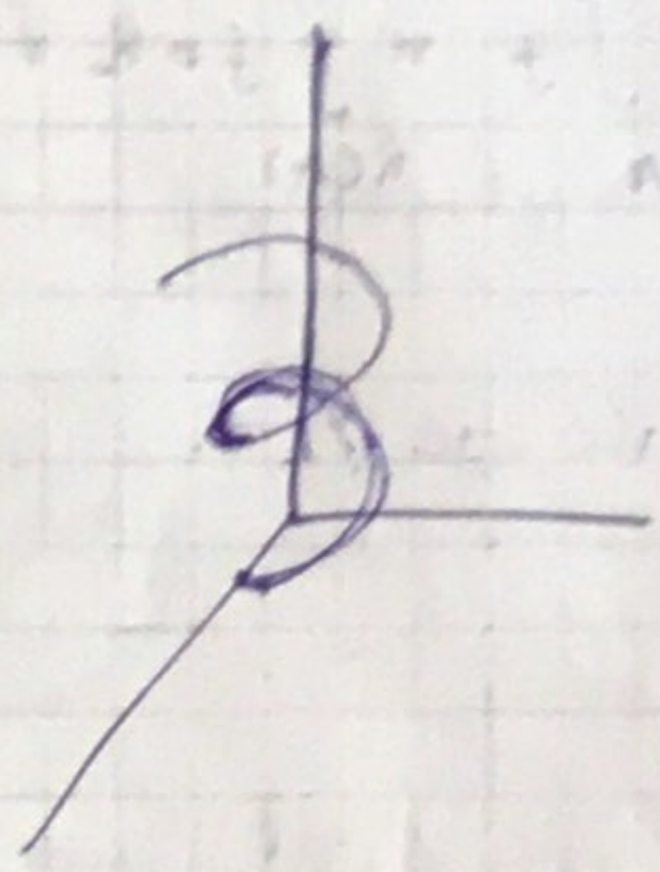


b) Analogno

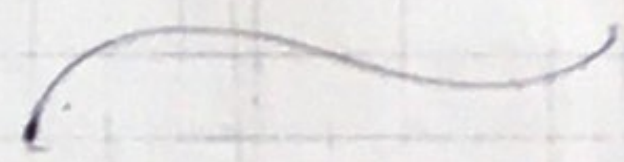
$g(t) = (R\cos t, R\sin t, ct)$, $t \in (0, 2\pi)$

$\|g'(t)\| = \sqrt{R^2 + c^2}$

$\int_0^{2\pi} \|g(t)\| dt = \sqrt{R^2+c^2} \cdot 2\pi$



$x \xrightarrow{g} (x, f(x))$



$g' = (1, f')$

$\|g'\| = \sqrt{1 + f'^2}$